Statistical mechanics of a multiconnected Hopfield neural-network model in a transverse field

Yu-qiang Ma

Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China;
Center for Advanced Studies in Science and Technology of Microstructures, Nanjing 210093, China;
*and Department of Physics, Nanjing University, Nanjing 210093, China**

Chang-de Gong

Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China
and Department of Physics, Nanjing University, Nanjing 210093, China

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The Hopfield neural-network model with p-spin interactions in the presence of a transverse field is introduced and solved exactly in the limit $p\to\infty$. In the phase diagrams drawn as a function of the temperature, the important results such as reentrance are found, and the effects of the quantum fluctuations on the phase transitions, the retrieval phase, and the storage ratio α are examined.

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In recent years there has been much interest in models of neural networks, which attempt to explain intriguing features such as memory, learning, and information storage and retrieval in terms of the collective properties of neural networks [1]. In particular, the Hopfield neural-network model [2] has been investigated extensively by using statistical mechanics, and most of the properties of this model are fairly well understood [1,3]. A natural generalization of this model of associative memory is the neural-network model with p-spin interactions [4–6], which is reminiscent of a p-spin interaction spin-glass model [7,8], and the large-p limit of the model is exactly soluble [9,10] without using replicas.

In a previous paper [11], we have discussed the effects of a transverse field Γ on the phase diagram of a Hopfield model with Hamiltonian H,

$$H=-\sum_{i,j}^{N}J_{ij}S_i^zS_j^z-\Gamma\sum_i S_i^x\ , \tag{1}$$

where S_i^z and S_i^x are Pauli spin matrices at ith spin and N is the total number of neurons. The synaptic couplings J_{ij} are determined by the Hebb rule

$$J_{ij}=N^{-1}\sum_{\mu=1}^{\alpha N}\xi_i^\mu\xi_j^\mu\ , \tag{2}$$

which corresponds to the situation that αN patterns $\{\xi^1,\xi^2,\ldots,\xi^{\alpha N}\}$ are stored. In this assignment every ξ_i^μ is regarded as an independently distributed stochastic variable, taking the values ±1 with equal probabilities. Our motivation to approach this quantum problem is to consider the tunneling effects among the neurons (represented by a transverse field Γ), and is a theoretical construct which introduces quantum effects to a classical problem in a natural way. In this paper we consider a transverse multiconnected Hopfield model which is the quantum analog of the neural-network models with p-spin interactions. We will be interested in particular in the $p\to\infty$ limit, where the thermodynamics of the model can be solved exactly. The effects of quantum fluctuations on phase diagrams of the model are examined.

We consider the model with the Hamiltonian

$$H=-\sum_{i_1<\cdots<i_p}J_{i_1,i_2,\ldots,i_p}S_{i_1}^zS_{i_2}^z\cdots S_{i_p}^z-\Gamma\sum_i S_i^x\ . \tag{3}$$

Here the couplings $J_{i_1,i_2,\ldots,i_p}$ are given by a generalized Hebb rule [4],

$$J_{i_1,i_2,\ldots,i_p}=\frac{p!}{N^{p-1}}\sum_{\mu=1}^{n}\xi_{i_1}^\mu\xi_{i_2}^\mu\cdots\xi_{i_p}^\mu\ , \tag{4}$$

where the number of patterns is $n=2N^{p-1}\alpha/p!$, and α is the storage ratio. When $\Gamma=0$, this model is identical to the generalization of the Hopfield model introduced by Gardner [4]. For $p=2$ and $\Gamma=0$ the model reduces to the Hopfield neural-network model.

As usual, we introduce the effective Hamiltonian for the ith spin,

$$H_i=-h_iS_i^z-\Gamma S_i^x\ , \tag{5}$$

where the local field $h_i=\sum_{i_2<\cdots<i_p}J_{i,i_2,\ldots,i_p}m_{i_2}m_{i_3}\cdots m_{i_p}$ and m_i is the local magnetization at site i. After diagonalizing H_i the partition function of a single spin is easily obtained,

$$Z=\mathrm{Tr}\exp(-\beta H_i)=2\cosh\beta(h_i^2+\Gamma^2)^{1/2}\ , \tag{6}$$

where β is the inverse temperature. The corresponding magnetization per spin then reads

$$m_i(h_i,\Gamma)=h_i(h_i^2+\Gamma^2)^{-1/2}\tanh\beta(h_i^2+\Gamma^2)^{1/2}\ . \tag{7}$$

The overlap m^μ between stored patterns $\{\xi_i^\mu\}$ and system state $\{m_i\}$ is defined by

$$m^\mu=N^{-1}\sum_i\xi_i^\mu m_i(h_i,\Gamma)\ . \tag{8}$$

For convenience we consider, as usual, only a pattern $\{\xi_i^1\}$ to be condensed and the remaining $(p-1)$ patterns to have an overlap of at most order $O(1/\sqrt{N})$. This

*Address for correspondence.

retrieval-state solution is described by the order parameter m^1,

$$m^1 = N^{-1} \sum_i \xi_i^1 m_i(h_i, \Gamma) , \tag{9}$$

and the random overlap correlation is measured by the order parameter r,

$$r = \frac{1}{\alpha} \sum_{\mu>1}^{n} \left[N^{-1} \sum_i \xi_i^\mu m_i(h_i, \Gamma) \right]^2 . \tag{10}$$

In the thermodynamic limit $N \to \infty$, Eq. (9) can be written as an integral over the distribution of local fields $P(H^1)$,

$$m^1 = \int dH^1 P(H^1) m_i(H^1, \Gamma) , \tag{11}$$

where $H_i^1 = \xi_i^1 h_i$. The local alignment field H_i^1 on site i can now be written as

$$\begin{aligned} H_i^1 &= \frac{p!}{N^{p-1}} \xi_i^1 \sum_{\mu=1}^{n} \sum_{i_2 < \cdots < i_p} \xi_i^\mu \xi_{i_2}^\mu \cdots \xi_{i_p}^\mu m_{i_2} m_{i_3} \cdots m_{i_p} \\ &= p(m^1)^{p-1} + \frac{p!}{N^{p-1}} \xi_i^1 \sum_{\mu>1}^{n} \xi_i^\mu \sum_{i_2 < \cdots < i_p} \xi_{i_2}^\mu \xi_{i_3}^\mu \cdots \xi_{i_p}^\mu m_{i_2} \\ &\quad \times m_{i_3} \cdots m_{i_p} . \end{aligned} \tag{12}$$

The right-hand side of Eq. (12) can be divided into two parts: a signal term coming from the contribution of pattern 1 to $J_{i,i_2,\ldots,i_p}$ which is equal to $p(m^1)^{p-1}$ and therefore favors its recall and a noise term coming from the contribution to $J_{i,i_2,\ldots,i_p}$ of all other patterns which has mean zero and Gaussian fluctuations of variance σ^2,

$$\sigma^2 = n \left[\frac{p!}{N^{p-1}} \right]^2 \frac{1}{(p-1)!} \left[\sum_i m_i^2 \right]^{p-1} = 2\alpha p q^{p-1} , \tag{13}$$

where $q \equiv N^{-1} \sum_i m_i^2$ is the usual spin-glass order parameter. Thus the distribution $P(H^1)$ for the local fields is given by the normalized Gaussian distribution

$$P(H^1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-[H^1 - p(m^1)^{p-1}]^2}{2\sigma^2} \right\} , \tag{14}$$

where the variance σ^2 is given by (13). Finally, the self-consistent equations for the overlap m^1 and the spin-glass order parameter q are given by the following integrals over the distribution of local fields $P(H^1)$:

$$\begin{aligned} m^1 &= \int \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \frac{p(m^1)^{p-1} + \sqrt{\alpha r} y}{\{\Gamma^2 + [p(m^1)^{p-1} + \sqrt{\alpha r} y]^2\}^{1/2}} \\ &\quad \times \tanh\beta\{\Gamma^2 + [p(m^1)^{p-1} + \sqrt{\alpha r} y]^2\}^{1/2} , \end{aligned} \tag{15}$$

$$\begin{aligned} q &= \int \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \frac{[p(m^1)^{p-1} + \sqrt{\alpha r} y]^2}{\{\Gamma^2 + [p(m^1)^{p-1} + \sqrt{\alpha r} y]^2\}} \\ &\quad \times \tanh^2\beta\{\Gamma^2 + [p(m^1)^{p-1} + \sqrt{\alpha r} y]^2\}^{1/2} , \end{aligned} \tag{16}$$

$$r = 2pq^{p-1} . \tag{17}$$

The phase diagram of the model can be obtained from Eqs. (15)–(17), and the result reduces to those for the generalized Hopfield neural-network model discussed by Gardner [4] when $\Gamma = 0$. Without recourse to replicas, approximate values of the maximum storage ratio α_c as a function of p can be obtained easily along the lines analyzed by Gardner [4]. However, we will restrict ourselves in this paper to the $p \to \infty$ limit. In this limit, we consider only retrieval states which are either fully correlated with a specific input pattern 1 ($m^1 = 1, m^\mu = 0$ for $\mu \neq 1$) or states which are uncorrelated with any pattern ($m^\mu = 0$ for all μ). This is because $m^\mu \leq 1$ and it follows that $(m^\mu)^p = 1$ if $m^\mu = 1$ and $(m^\mu)^p = 0$ otherwise. Similarly, r can either be ∞ (if $q = 1$) or 0 (otherwise). The resulting phase diagram of the model is shown in Fig. 1. We find that even an arbitrarily weak transverse field destroys the spin-glass transitions which exist in the classical multiconnected Hopfield model [4]. This result is also in contrast to the Hopfield neural-network model in a transverse field [11] where the weak transverse field does not destroy completely the onset of spin-glass freezing transition. Quantum fluctuations have the effect of shrinking the retrieval phase. With the increase of the strength of transverse field Γ, the regions of the retrieval phases decrease, and when $\Gamma = 1$, the retrieval phases disappear. Another interesting feature of the retrieval phases in the presence of transverse fields Γ (see Fig. 1) is that they all show reentrant paramagnet behavior below a certain tem-

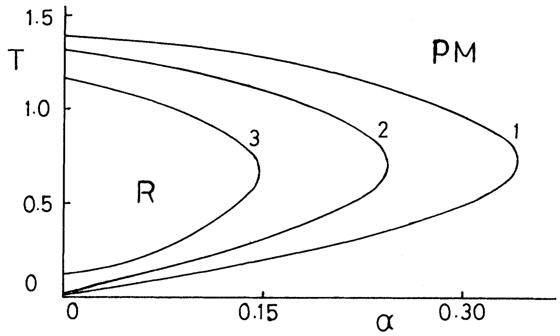

FIG. 1. The phase diagram of the systems in the large-p limit for three values of Γ. PM denotes the paramagnetic phase, R the retrieval phase. Curves 1, 2, and 3 correspond to $\Gamma = 0.2$, 0.5, and 0.7, respectively.

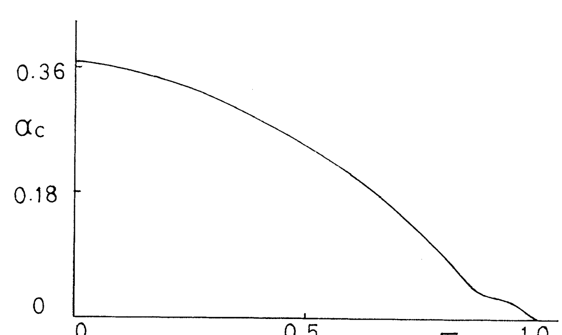

FIG. 2. Values of the maximum storage ratio α_c as a function of Γ.

perature contrary to that found for the phase diagram of the quantum Hopfield model in a transverse field [11]. This reentrant behavior may be attributed to the competition between quantum effects and random-pattern-induced fluctuations. The maximum value of storage ratio α_c is obtained at a certain nonzero temperature which is located above the zero temperature critical value $\alpha_c(T=0)$ for the classical cases. This obviously implies that the network can store more patterns at finite temperatures than at zero temperature. Figure 2 shows the dependence of the maximum storage ratio α_c on the transverse field Γ.

So far, the multiconnected Hopfield neural-network model has been generalized to the quantum case and solved by using a simple method without the use of replicas. The phase diagram is obtained in the limit $p \to \infty$, and the effects of quantum fluctuations on the phase transitions are examined. Our conclusion is that the transverse field, even if it is arbitrarily weak, destroys the spin-glass phase transitions in multiconnected Hopfield models when $p \to \infty$. Reentrant phase transitions, which may be caused by the competition between quantum effects and the random overlap correlations, occur with nonzero values of transverse fields.

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